

# Constants of motion in stationary axisymmetric gravitational fields

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## ABSTRACT

The motion of a test particle in a stationary axisymmetric gravitational field is generally nonintegrable unless, in addition to the energy and angular momentum about the symmetry axis, an extra nontrivial constant of motion exists. We use a direct approach to systematically search for a nontrivial constant of motion polynomial in the momenta.

By solving a set of quadratic integrability conditions, we establish the existence and uniqueness of the family of stationary axisymmetric Newtonian potentials admitting a nontrivial constant quadratic in the momenta. Although such constants do not arise from a group of diffeomorphisms, they are Noether-related to symmetries of the action and associated with irreducible rank-2 Killing-Stäckel tensors. The multipole moments of this class of potentials satisfy a “no-hair” recursion relation  $M_{2l+2} = a^2 M_{2l}$  and the associated quadratic constant is the Newtonian analogue of the Carter constant in a Kerr-de Sitter spacetime.

We further explore the possibility of invariants quartic in the momenta associated with rank-4 Killing-Stäckel tensors and derive a new set of quartic integrability conditions. We show that a subset of the quartic integrability conditions are satisfied by potentials whose even multipole moments satisfy a generalized “no-hair” recursion relation  $M_{2l+4} = (a^2 + b^2)M_{2l+2} - a^2 b^2 M_{2l}$ . However, the full set of quartic integrability conditions cannot be satisfied nontrivially by any Newtonian stationary axisymmetric vacuum potential. We thus establish the nonexistence of irreducible invariants quartic in the momenta for motion in such potentials.

**Key words:** gravitation – black hole physics – celestial mechanics – chaos – stellar dynamics – galaxies: star clusters

## 1 INTRODUCTION

Since the work of Euler, Lagrange and Jacobi, the motion of a test particle in a Newtonian dipole field<sup>1</sup> has been known to be completely integrable in terms of quadratures. In addition to energy and angular momentum about the symmetry axis, there exists a third nontrivial constant of motion quadratic in the momenta. The constant was first discovered by Euler (1760, 1764) for two-dimensional (meridional) motion, and the problem is known as the *Euler problem* (Lukyanov, Emeljanov, & Shirmin 2005). Lagrange (1766) extended Euler’s solution to three-dimensional motion and made a further generalization by allowing a Hookian (spring)

center to be included between the two Newtonian (gravitational) centers. This generalization is known as the *Lagrange problem* (cf. Lukyanov et al. 2005). The problem was further studied by Jacobi (2009) using separation of variables of the Hamilton-Jacobi equation in prolate spheroidal coordinates.

The literature regarding the problem of two fixed centers is surveyed by Lukyanov et al. (2005) and the orbits are studied in Ó’Mathúna (2008). The quadratic<sup>2</sup> constant has been studied by several authors using the Hamilton-Jacobi approach (Stäckel 1890; Whittaker 1989; Eddington 1915; Kuzmin 1956; Lynden-Bell 1962; de Zeeuw 1985a,b,c). If one considers the distance between the two centers to be imaginary, then one obtains a potential separable in oblate spheroidal coordinates. This is known in satellite

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<sup>1</sup> Following terminology in Misner et al. (1973), a Newtonian dipole will refer to a pair of fixed positive-mass centers each of which creates a Newtonian gravitational field

<sup>2</sup> A quadratic constant will refer to a constant quadratic in the momenta

geodesy as the *Vinti potential* and has been used to approximate the gravitational field around the oblate earth (Vinti 1960, 1963, 1969, 1971; Vinti et al. 1998). A further generalization is possible by relaxing equatorial plane symmetry. In the oblate case, this is accomplished by considering two Newtonian fixed centers with complex-conjugated masses located at constant imaginary distance. The resulting *Darboux–Gredeaks potential* has been used to approximate the gravitational field around other oblate planets (Aksenov, Grebenikov, & Demin 1963, Lukyanov et al. 2005).

Lynden-Bell (2003) has provided a simple and elegant derivation of the quadratic constant in the Euler problem, by noting that the kinetic part of the constant is the dot product of the angular momenta about the two fixed centers of attraction. He also provided a generalization of the constant for a class of potentials that satisfy a certain integrability condition. It will be demonstrated in section 2.6 that this class is essentially that of the Lagrange problem, that is, it amounts to the addition of a Hooke term to the potential.

Israel (1970) and Keres (1967) have demonstrated that the dipole field of the Euler problem can be regarded as the Newtonian analogue of the Kerr solution in general relativity. Following Misner’s suggestion to seek analogues of the quadratic constant in Newtonian dipole fields, Carter discovered his quadratic constant of motion in Kerr spacetime by separation of variables (Carter 1968, 1977; Misner, Thorne, & Wheeler 1973). The analogy has been further elucidated by Lynden-Bell (2003), Flanagan & Hinderer (2007) and Will (2009).

Carter (1968, 2009b, 2010) also discovered a generalization of the Kerr solution with a non-zero cosmological constant, describing a rotating black hole in four dimensional de Sitter (or anti de Sitter) backgrounds. Carter’s quadratic constant of motion exists for this class of spacetimes as well and has been used to solve for the orbits in terms of hypergeometric functions (Kraniotis 2004, 2005, 2011). It will be shown in section 2.6 that the quadratic constant of the Lagrange two-center problem is the Newtonian analogue of the Carter constant in a Kerr-de Sitter spacetime.

The existence of the Carter constant is very useful for analyzing the orbit of a small black hole around a massive black hole. Detecting gravitational waves from these extreme mass ratio inspirals is a prime goal of the proposed New Gravitational Wave Observatory, eLISA/NGO (Jennrich et al. 2011; Amaro et al. 2012a,b). The influence of gravitational radiation reaction on the evolution of the Carter constant has been used to obtain the gravitational waveforms from orbits around Kerr black holes. As demonstrated by Ryan (1995), by measuring the mass, spin and at least one more non-trivial moment of the gravitational field of a black hole candidate via gravitational wave observations, one can test the validity of the no-hair theorem. This has raised interest in parametrizing spacetimes with multipole moments that deviate from those of the Kerr solution (Glampedakis & Babak 2006; Gair et al. 2008; Vigeland 2010; Vigeland & Hughes 2010; Vigeland et al. 2011; Johannsen & Psaltis 2010; Apostolatos et al. 2009; Lukes-Gerakopoulos et al. 2010; Contopoulos et al. 2011). Such deviations generally lead to loss of the Carter constant. Whether there exist stationary axisymmetric space-

times more general than Kerr-de Sitter that admit such a constant is an open question (Brink 2008a,b, 2010a,b, 2011; Mirshekari & Will 2010; Kruglikov & Matveev 2011; Lukes-Gerakopoulos 2012).

The aim of this work is to answer the analogous question in Newtonian gravity. We use a direct approach (Hietarinta 1987) to systematically search for a nontrivial constant polynomial in the momenta. This approach consists of directly solving the Killing equations and certain integrability conditions. In section 2 we show that the constant in the Euler and Lagrange two-center problems is the *unique quadratic constant* for motion around a stationary axisymmetric massive object with equatorial reflection symmetry. In section 2.7 this constant is shown to be Noether-related. In section 3 we consider the next natural generalization, a constant *quartic* in the momenta. Using the direct approach, we show that *no such constant exists* for stationary axisymmetric vacuum potentials in Newtonian gravity. Conclusions are given in section 4.

## 2 INVARIANTS QUADRATIC IN THE MOMENTA

### 2.1 Killing equation and integrability conditions

The motion of a test particle in a Newtonian gravitational field  $\Phi$  is independent of the particle mass, so for simplicity we can set the latter equal to unity. This motion is described by an action functional

$$S[x, p] = \int_{t_1}^{t_2} dt [p_i \dot{x}^i - H(x, p)] \quad (1)$$

where position  $x^i$  and momentum  $p_i$  are treated as independent variables,

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j + \Phi(x) \quad (2)$$

is the Hamiltonian,  $g^{ij}$  is the inverse of the Euclidian metric  $g_{ij}$  in  $\mathbb{E}^3$  and summation over repeated indices is implied. Indices are raised and lowered with this metric throughout the paper. Unless otherwise noted, we will be using Cartesian coordinates, so that  $g_{ij} = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta. The Cartesian components of the canonical momentum  $p_i = g_{ij} \dot{x}^j$  will then coincide with those of the kinetic momentum  $\dot{x}^i$ .

Spatial translations and rotations in  $\mathbb{E}^3$  are generated by the vector fields

$$\mathbf{X}_k = \mathbf{e}_k, \quad \mathbf{R}_i = \varepsilon^k_{ji} x^j \mathbf{e}_k \quad (3)$$

where  $\mathbf{e}_k = \{\hat{x}, \hat{y}, \hat{z}\}$  are Cartesian basis vectors and  $\varepsilon_{ijk}$  is the Levi-Civita tensor. We are interested in orbits of a test particle in the vicinity of a *stationary* and *axisymmetric* massive object, an object whose gravitational potential satisfies

$$\partial_t \Phi = 0 \quad (4)$$

$$\mathcal{L}_\varphi \Phi = \varphi^k \partial_k \Phi = (x \partial_y - y \partial_x) \Phi = 0 \quad (5)$$

Here,  $\mathcal{L}_\varphi$  denotes the Lie-derivative along the Killing vector field

$$\varphi \equiv \mathbf{R}_z = x \hat{y} - y \hat{x} \quad (6)$$

that generates rotations about the  $z$ -axis (the axis of symmetry). By virtue of Noether's theorem, invariance with respect to time translations and azimuthal rotations implies respectively conservation of the Hamiltonian  $H$  and the component

$$L_z = \varphi^i p_i = R_z^i p_i = x p_y - y p_x \quad (7)$$

of angular momentum. Rotations about the  $x$  and  $y$  axes, generated by the vector fields  $\mathbf{R}_x$  and  $\mathbf{R}_y$  are *not* considered symmetries of the problem, so the components  $L_x = R_x^i p_i$  and  $L_y = R_y^i p_i$  of angular momentum are *not* generally conserved. For example, the potential around a rotating massive object (such as a star or planet) in hydrostationary equilibrium, may be expected to be axisymmetric, but not spherically symmetric, due to the rotationally induced deformation. Without further symmetries,  $H$  and  $L_z$  are the only independent integrals of motion and, since the motion of test particles is three-dimensional, the problem is in general non-integrable. Nevertheless, one may seek special types of stationary axisymmetric potentials that admit a third non-trivial constant of motion and are therefore integrable.

If one seeks a constant of motion linear in the momenta, then one is quickly led to rotational or translational symmetries as the only choices, which have been already exhausted as explained above. The next natural step is to seek potentials admitting a nontrivial constant of motion *quadratic* in the momenta,

$$I(x, p) = K^{ij}(x) p_i p_j + K(x), \quad (8)$$

where the symmetric tensor  $K^{ij}$  and the scalar  $K$  are functions of position. The above quantity is conserved iff it commutes with the Hamiltonian, in the sense of a vanishing Poisson bracket:

$$\frac{dI}{dt} = \{I, H\} \equiv \frac{\partial I}{\partial x^k} \frac{\partial H}{\partial p_k} - \frac{\partial I}{\partial p_k} \frac{\partial H}{\partial x^k} = 0. \quad (9)$$

Substituting the Hamiltonian (2) and the ansatz (8) into the above Poisson bracket yields

$$\{I, H\} = \partial^k K^{ij} p_i p_j p_k + (\partial^j K - 2K^{jk} \partial_k \Phi) p_j \quad (10)$$

where  $\partial_k = \frac{\partial}{\partial x^k}$ . In order that this Poisson bracket vanish for all orbits, the following necessary and sufficient conditions (Boccaletti & Pucacco 2003) must be satisfied:

$$\partial^k K^{ij} = 0 \quad (11)$$

$$\partial^j K = 2K^{jk} \partial_k \Phi, \quad (12)$$

where parentheses denote symmetrization over the enclosed indices; that is,  $A^{(ij)} = \frac{1}{2}(A^{ij} + A^{ji})$ . Eq. (11) is a Killing equation in Euclidian space; a solution  $K^{ij}$  to the above equations will be referred to as a Killing-Stäckel tensor (Carter 1977). From Eq. (12), we obtain a necessary, but not sufficient, integrability condition

$$\partial^{[i} (K^{j]k} \partial_k \Phi) = \frac{1}{2} \partial^{[i} \partial^{j]} K = 0 \quad (13)$$

where square brackets denote antisymmetrization over the enclosed indices, that is  $A^{[ij]} = \frac{1}{2}(A^{ij} - A^{ji})$ .

Our assumptions of stationarity and axisymmetry already guarantee the existence of two independent solutions to the above set of equations. First, the metric itself is already a Killing-Stäckel tensor, because equations (11)–(13)

are satisfied by  $K^{ij} = \frac{1}{2}g^{ij} = \frac{1}{2}\delta^{ij}$ ,  $K = \Phi$ , and the associated conserved quantity is simply the Hamiltonian (2). Secondly, the above equations are also satisfied by the reducible Killing-Stäckel tensor  $K^{ij} = \varphi^i \varphi^j$ , with  $K = 0$ , implying conservation of the quantity  $L_z^2$ . This second case is reducible to the linear invariant (7) associated with the axial Killing vector  $\varphi^i$ . We now explore the possibility of a third independent solution that can render the problem integrable.

The conditions (11)–(13) suggest a systematic method (Hietarinta 1987) for obtaining integrable potentials and the associated invariants of motion:

- (i) The tensor  $K^{ij}$  may be computed by solving Eq. (11) subject to the symmetries of the problem.
- (ii) With this tensor known, the integrability condition (13) provides a key restriction on the Newtonian potential  $\Phi$ . One may solve this condition (e.g. via a multipole method) to obtain a family of integrable potentials.
- (iii) Given a family of potentials  $\Phi$  that satisfy this integrability condition, one may obtain the scalar function  $K$  by integrating the components of Eq. (12).

With this method as the basis of our analysis, we proceed to carry out the prescribed steps in more detail.

## 2.2 Rank-two Killing-Stäckel tensors

One may straightforwardly solve Eq. (11) by noticing that the solution must be polynomial in the Cartesian coordinates. This is easily seen in one or two dimensions (c.f. Appendix A) and can be generalized to arbitrary dimensions and tensor rank (Horwood 2008). In three dimensions, Eq. (11) constitutes an overdetermined system of ten equations for the six independent components of the symmetric tensor  $K^{ij}$ . As shown by Horwood (2008), the most general solution to this system is a sum of symmetrized products of the translational and rotational vectors (3):

$$\mathbf{K} = K^{ij} \mathbf{X}_i \otimes \mathbf{X}_j = \mathcal{A}^{ij} \mathbf{X}_i \otimes \mathbf{X}_j + 2\mathcal{B}^{ij} \mathbf{X}_i \otimes \mathbf{R}_j + \mathcal{C}^{ij} \mathbf{R}_i \otimes \mathbf{R}_j \quad (14)$$

The components  $K^{ij}$  of the tensor  $\mathbf{K}$  are polynomial of second order in the Cartesian coordinates  $x^i$  and can be written as

$$K^{ij} = \mathcal{A}^{ij} + 2\mathcal{B}^{k(i} \varepsilon^{j)}_{kl} x^l + \mathcal{C}^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} x^k x^l \quad (15)$$

where  $\mathcal{A}^{ij} = \mathcal{A}^{(ij)}$ ,  $\mathcal{C}^{ij} = \mathcal{C}^{(ij)}$  are *symmetric*  $3 \times 3$  constant matrices and  $\mathcal{B}^{ij}$  is a *nonsymmetric*  $3 \times 3$  constant matrix. While a superficial counting results in 21 independent coefficients, the actual number of independent coefficients is 20. This is because the main diagonal elements of  $\mathcal{B}^{ij}$  appear in  $K^{ij}$  only in the combinations  $\mathcal{B}^{xx} - \mathcal{B}^{yy}$ ,  $\mathcal{B}^{yy} - \mathcal{B}^{zz}$ ,  $\mathcal{B}^{zz} - \mathcal{B}^{xx}$  and the sum of these three terms is zero.

## 2.3 Isometries

One may considerably reduce the number of unknown coefficients by imposing known symmetries of the action functional  $S$  on the orbital invariant  $I$ .

- (i) *Stationarity* corresponds to invariance of  $S$  under the group action of  $(\mathbb{R}, +)$  which represents time translations  $t \rightarrow t + \delta t$ . This symmetry has already been taken into

account, since all quantities have no explicit dependence on time  $t$  and an additive term  $\partial I/\partial t$  has been set to zero in Eq. (9).

(ii) *Axisymmetry* corresponds to invariance of  $S$  under the group action of  $\text{SO}(2)$  which represents infinitesimal rotations  $\{x, y, z\} \rightarrow \{x + y \delta\varphi, y - x \delta\varphi, z\}$  about the  $z$ -axis. This symmetry is generated by the Killing vector (6) and may be imposed by requiring

$$\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x}\right) I = 0 \quad (16)$$

With  $I$  given by Eq. (8), the above condition can be shown to be equivalent to the requirement that the functions  $K^{ij}$  and  $K$  remain unchanged under rotations about the  $z$ -axis:

$$\mathcal{L}_\varphi K^{ij} = \varphi^k \partial_k K^{ij} - K^{kj} \partial_k \varphi^i - K^{ki} \partial_k \varphi^j = 0 \quad (17)$$

$$\mathcal{L}_\varphi K = \varphi^k \partial_k K = 0 \quad (18)$$

Evidently, the symmetries (16), (17), (18) are inherited from the Hamiltonian, which is seen by the fact that these equations are also satisfied by replacing  $I, K^{ij}, K$  with  $H, \frac{1}{2}\gamma^{ij}, \Phi$  in the above three equations respectively. Substituting the general solution (15) into Eq. (17) yields the axisymmetry constraints

$$\mathcal{A}^{xy} = \mathcal{A}^{xz} = \mathcal{A}^{yz} = \mathcal{A}^{xx} - \mathcal{A}^{yy} = 0 \quad (19)$$

$$\begin{aligned} \mathcal{B}^{xz} &= \mathcal{B}^{zx} = \mathcal{B}^{yz} = \mathcal{B}^{zy} \\ &= \mathcal{B}^{xx} - \mathcal{B}^{yy} = \mathcal{B}^{xy} + \mathcal{B}^{yx} = 0 \end{aligned} \quad (20)$$

$$\mathcal{C}^{xy} = \mathcal{C}^{xz} = \mathcal{C}^{yz} = \mathcal{C}^{xx} - \mathcal{C}^{yy} = 0 \quad (21)$$

(iii) Further simplification is possible by assuming *equatorial plane reflection symmetry*. This corresponds to invariance of  $S$  under the discrete group action of  $\mathbb{Z}_2$  which represents reflections  $\{x, y, z, p_x, p_y, p_z\} \rightarrow \{x, y, -z, p_x, p_y, -p_z\}$  about the equatorial plane. (This assumption is not necessary for integrability, c.f. Lynden-Bell (2003), but we retain it for simplicity). Imposing this symmetry on the invariant (8) leads to the constraints

$$\mathcal{A}^{xz} = \mathcal{A}^{yz} = 0 \quad (22)$$

$$\mathcal{B}^{xx} = \mathcal{B}^{yy} = \mathcal{B}^{zz} \quad (23)$$

$$\mathcal{B}^{xy} = \mathcal{B}^{yx} = 0 \quad (24)$$

$$\mathcal{C}^{xz} = \mathcal{C}^{yz} = 0 \quad (25)$$

Note that the constraints (19)–(21) and (22)–(25) are independent, since axisymmetry and reflection symmetry are separate assumptions. Imposing these two sets of constraints on the tensor (15) yields

$$\begin{aligned} \mathbf{K} &= \mathcal{C}^{yy} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \\ &+ (\mathcal{A}^{zz} - \mathcal{A}^{yy}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{A}^{yy} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ (\mathcal{C}^{zz} - \mathcal{C}^{yy}) \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (26)$$

This is the most general solution to Eq. (11) consistent with the symmetry  $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$ . For clarity, we set  $\kappa \equiv \mathcal{C}^{yy}$ ,  $\lambda \equiv \mathcal{A}^{zz}$ ,  $\mu \equiv \mathcal{C}^{zz} - \mathcal{C}^{yy}$  and  $a \equiv \sqrt{(\mathcal{A}^{zz} - \mathcal{A}^{yy})/\mathcal{C}^{yy}}$ . The parameter  $a$  is allowed to be real or

imaginary and will be shown to depend on the gravitational source. (This reparametrization entails no loss of generality, as the sign of  $(\mathcal{A}^{zz} - \mathcal{A}^{yy})/\mathcal{C}^{yy}$  is unrestricted. We have also excluded the possibility of vanishing  $\mathcal{C}^{yy}$  with nonvanishing  $(\mathcal{A}^{zz} - \mathcal{A}^{yy})$ . This case would lead to a Killing-Stäckel tensor  $(\mathcal{A}^{zz} - \mathcal{A}^{yy})\delta_z^i \delta_z^j$ , reducible to a Killing vector  $\delta_z^i$  that generates translations along the  $z$ -axis. But this is *not* a symmetry of the problem by assumption, i.e. we have excluded cylindrical symmetry.) The above solution may then be written as

$$K^{ij} = \kappa A^{ij} + \lambda g^{ij} + \mu \varphi^i \varphi^j \quad (27)$$

where  $g_{ij} = \delta_{ij}$  is the Euclidian metric,  $\varphi$  is the axial Killing vector (6),

$$\begin{aligned} A^{ij} &= \delta^{kl} R_k^i R_l^j + a^i a^j - a^2 g^{ij} \\ &= (r^2 g^{ij} - x^i x^j) - (a^2 g^{ij} - a^i a^j) \\ &= g^{mn} \varepsilon_{km}^i \varepsilon_{ln}^j (x^k x^l - a^k a^l) \\ &= g^{mn} \varepsilon_{km}^i \varepsilon_{ln}^j (x^k + a^k)(x^l - a^l) \end{aligned} \quad (28)$$

is a new nontrivial Killing-Stäckel tensor,  $r = \sqrt{x^k x_k}$  is a radial coordinate and  $a^i = a \delta_z^i$  are the components of the vector  $\mathbf{a} = a \hat{\mathbf{z}}$ . With  $\kappa, \lambda, \mu$  regarded as arbitrary coefficients, the general solution (27) is a linear combination of three independent solutions: the known Killing-Stäckel tensors  $\gamma^{ij} = \delta^{ij}$  and  $\varphi^i \varphi^j$ , associated with stationarity and axisymmetry, and a third independent solution  $A^{ij}$  associated with a Noether symmetry to be discussed later. The above expression gives the *most general rank-two Killing tensor* in stationary-axisymmetric vacuum potentials. This completes step (i) of the prescribed procedure.

## 2.4 Integrability condition: no-hair relation

Although the solution (28) satisfies Eq. (11) and the symmetries of the problem, it does not lead to a conserved quantity unless the condition (12) is satisfied. Our next step is thus to use the integrability condition (13) of Eq. (12) to obtain a family of potentials  $\Phi$  for which  $A^{ij}$  leads to a conserved quantity. The Newtonian gravitational field around an axisymmetric object is completely characterized by a set of mass multipole moments  $\{M_L\}$ , by means of the expansion

$$\Phi = - \sum_{L=0}^{\infty} \frac{M_L}{r^{L+1}} P_L(z/r) \quad (29)$$

where  $P_L$  are the Legendre polynomials. This expansion is consistent with a stationary axisymmetric vacuum potential that vanishes at infinity.

If one also requires equatorial plane reflection symmetry, then the potential  $\Phi$  must be an odd function of  $z$ , so the odd moments  $M_1, M_3, \dots$  must vanish. Then, substituting  $K^{ij} = A^{ij}$  and  $\Phi$ , as given by eqs. (28) and (29), into the integrability condition (13), we find by straightforward algebra that the latter is satisfied *if and only if* the even multipole moments satisfy the recursion relation:

$$M_{L+2} = a^2 M_L \quad (L = 0, 2, \dots) \quad (30)$$

or, equivalently,

$$M_L = a^L M_0 \quad (L = 0, 2, \dots) \quad (31)$$

where  $a^L$  denotes the  $L$ -th power of  $a$  in the above two



equations. This relation is analogous to the “no-hair” relation for Kerr black holes (Israel 1970; Hansen 1974; Will 2009). All nonzero multipoles are determined by the mass  $M_0$  and the parameter  $a$ . If  $a$  is real (imaginary) then the quadrupole moment  $M_2 = a^2 M_0$  is positive (negative) and the expansion (29) describes the field of a prolate (oblate) object. Summing the Legendre series (29) using Eq. (31) yields (Trahanas 2004)

$$\Phi = -\frac{M_0/2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{M_0/2}{\sqrt{x^2 + y^2 + (z+a)^2}} \quad (32)$$

In the prolate case, the above potential is that of Euler’s three body problem: a test mass moving in the gravitational field of two point sources, each of mass  $M_0/2$ , fixed at positions  $\pm a \hat{z}$ .

In the oblate case, the replacement  $a \rightarrow i a$  (where  $i^2 = -1$ ) allows one to write the above potential as

$$\Phi = -\frac{M_0}{\rho + a^2 z^2 / \rho^3} \quad (33)$$

where  $\rho$  is an ellipsoidal coordinate defined by

$$\frac{x^2 + y^2}{\rho^2 + a^2} + \frac{z^2}{\rho^2} = 1 \quad (34)$$

Vinti (1960, 1963, 1969, 1971); Vinti et al. (1998) has shown that the above potential is the most general solution to the three dimensional Laplace equation that separates the Hamilton-Jacobi equation in oblate spheroidal coordinates. (A similar statement can be made for the potential (32) in prolate spheroidal coordinates). Vinti’s potential has been frequently used in satellite geodesy to approximate the gravitational field around the oblate earth.

We have so far shown that the solutions (28) and (32) are the unique stationary, axisymmetric, equatorially symmetric, vacuum solutions to eqs. (11) and (13). (Note that we have excluded the possibility of cylindrical symmetry as we are interested in the gravitational field of massive objects with compact support near the origin.) This completes step (ii) of our procedure.

## 2.5 Existence and uniqueness of the quadratic invariant

The last step towards obtaining an invariant is to obtain the scalar contribution  $K$  to the invariant (8). This is done by solving the condition (12) with  $K^{ij} = A^{ij}$  and  $\Phi$  given respectively by eqs. (28) and (29). The solution  $K = A$  is obtained by integrating the  $x$  component of Eq. (12) with respect to  $x$ , or the  $y$  component with respect to  $y$ . Up to some additive constant, we find

$$\begin{aligned} A &= 2 \int dx A^{ix} \partial_i \Phi = 2 \int dy A^{iy} \partial_i \Phi \\ &= \frac{M_0 a (z+a)}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0 a (z-a)}{\sqrt{x^2 + y^2 + (z-a)^2}} \end{aligned} \quad (35)$$

Finally, substituting Eq. (28) into (8) and using the Lagrange identity, the quadratic invariant is written

$$\begin{aligned} I &= A^{ij} p_i p_j + A \\ &= r^2 |\mathbf{p}|^2 - |\mathbf{x} \cdot \mathbf{p}|^2 - (a^2 |\mathbf{p}|^2 - |\mathbf{a} \cdot \mathbf{p}|^2) + A \\ &= |\mathbf{x} \times \mathbf{p}|^2 - |\mathbf{a} \times \mathbf{p}|^2 + A \\ &= [(\mathbf{x} + \mathbf{a}) \times \mathbf{p}] \cdot [(\mathbf{x} - \mathbf{a}) \times \mathbf{p}] + A \end{aligned} \quad (36)$$

with  $A$  given by Eq. (35). The above expression holds for the prolate case,  $a = |a|$ . In the oblate case,  $a = i|a|$ , the above expression still gives a true result. In the spherically symmetric limit,  $a \rightarrow 0$ , the above invariant reduces to  $I \rightarrow |\mathbf{x} \times \mathbf{p}|^2$  which is a natural consequence of conserved net angular momentum. The first integrals  $H, L_z, I$  are independent and in involution (that is,  $\{H, L_z\} = 0, \{H, I\} = 0$  and  $\{I, L_z\} = 0$ ). Thus, the system is Liouville-integrable.

We have shown that the potential (33) is the unique stationary, axisymmetric, equatorially symmetric, vacuum potential admitting a nontrivial constant of motion quadratic in the momenta, Eq. (36). The potential (33) can be considered the Newtonian analogue (Keres 1967; Israel 1970; Lynden-Bell 2003; Will 2009) of the Kerr solution in general relativity. The analogy is manifest when the Kerr metric is written in Kerr-Schild coordinates (Kerr & Schild 1965). The associated constant of motion is traditionally obtained by separating the Hamilton-Jacobi equation in Boyer-Lindquist coordinates (Carter 1968; Misner et al. 1973). It was in this way that Carter originally discovered his constant, following Misner’s suggestion to seek analogies to the constant (36) in Newtonian dipole fields. This analogy persists in the presence of a cosmological constant as demonstrated in the following section.

Another interesting property is the following: One may interchange the roles of the metric  $g_{ij}$  and potential  $\Phi$  with those of the Killing-Stäckel tensor  $A_{ij}$  and scalar field  $A$  respectively. Then the momentum map is generated by  $I = A^{ij} p_i p_j + A$  (which plays the role of the Hamiltonian) and the quadratic constant of motion is given by  $H = \frac{1}{2} g^{ij} p_i p_j + \Phi$ . This duality also exists in the relativistic case of the Kerr spacetime (Carter 2009a).

## 2.6 Cosmological constant: analogy with a Kerr-de Sitter spacetime

One may generalize the result of the previous section in various directions, by relaxing certain assumptions. The assumption that the motion is in vacuum may be relaxed by adding terms with a non-vanishing Laplacian to the multipole expansion (29) of the gravitational potential. For example, one may attempt to add a spherically symmetric power-law potential of the form  $r^l$ . Doing so, and repeating the previous steps, leads to the same expression (28) for the Killing tensor  $A^{ij}$  as before. Direct substitution shows that the integrability condition (13) is satisfied *if and only if* the moments  $\{M_L\}$  satisfy the no-hair relation (30) and the only power law term that can be added is proportional to  $r^2$ . This leads to a potential

$$\begin{aligned} \Phi &= -\frac{M_0/2}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0/2}{\sqrt{x^2 + y^2 + (z-a)^2}} \\ &\quad - \Lambda(x^2 + y^2 + z^2) \end{aligned} \quad (37)$$

In the prolate case, with  $a$  real, the above potential is that of two force centers, with Newton’s inverse square law of gravitational attraction supplemented by a linear cosmological constant (or Hooke) term. (Note that the last term in the above equation may be regarded as the contribution of a spring connecting the test particle to the origin or as the contribution of two springs connecting the test particle to the two centers at  $\pm a \hat{z}$ ). The associated constant of motion

is given by Eq. (36) with

$$A = \frac{M_0 a(z+a)}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0 a(z-a)}{\sqrt{x^2 + y^2 + (z-a)^2}} + 2\Lambda a^2(x^2 + y^2) \quad (38)$$

In the oblate case, with  $a$  imaginary, the last term may again be interpreted as the contribution of a cosmological constant. With the substitution  $a \rightarrow i a$ , the potential (37) can be naturally regarded as the Newtonian analogue of a Kerr-de Sitter spacetime and the constant (38) as the analogue of the Carter constant in that spacetime (Carter 1968). For this spacetime the metric has a  $tt$  component given by  $g_{tt} = -(1+2\Phi)$  in Kerr-Schild coordinates, with  $\Phi$  given by Eq. (37) and  $a$  replaced by  $i a$ .

Lynden-Bell (2003) provided a generalization of the quadratic constant (36) and the two-center potential (32). His result holds for potentials satisfying a certain integrability condition, which takes the form of a wave equation,  $(\partial^2/\partial r_1^2 - \partial^2/\partial r_2^2)(r_1 r_2 \Phi) = 0$  in his two-center coordinates  $\mathbf{r}_{1,2} = \mathbf{r} \pm \mathbf{a}$  (Lynden-Bell 2003, section 3). Assuming equatorial symmetry and substituting the multipole expansion (29) into this integrability condition, we recover the no-hair relation (29) and the potential (32). If we attempt to add power law terms of the form  $r_1^l, r_2^l$  to the potential  $\Phi$  we find that the only possibility is  $r_1^2 + r_2^2 = 2(r^2 + a^2)$ , giving rise to the potential (37) up to a constant. We infer that Lynden-Bell's generalization accounts for the presence of a cosmological constant term, which gives rise to the Newtonian analogue of the Kerr-de Sitter spacetime discussed above.

The direct approach employed here is more algorithmic and computationally intensive compared to other approaches (Lynden-Bell 2003; Will 2009). But the advantages of the direct approach are that it establishes uniqueness and that it can be straightforwardly generalized to higher order or relativistic invariants. We have shown that the unique stationary, axisymmetric, equatorially symmetric potential admitting an invariant quadratic in the momenta is given by Eq. (37). The assumption of equatorial symmetry may be relaxed (Lynden-Bell 2003) by dropping conditions (22)-(25), but we do not pursue this here.

## 2.7 Generalized Noether symmetry

The generalized Noether theorem may be stated as follows (Ioannou & Apostolatos 2004). Consider the  $\epsilon$ -family of infinitesimal transformations

$$x^i \rightarrow x_\epsilon^i = x^i + \epsilon K^i(x, \dot{x}, t) \quad (39)$$

which depend on position *and velocity*, for a small parameter  $\epsilon$ . If this family of transformations leaves the action  $S = \int L dt$  invariant, or, equivalently, changes the Lagrangian  $L$  by a total time derivative of some scalar  $K(x, t)$ ,

$$L \rightarrow L_\epsilon = L - \epsilon \frac{dK}{dt} \quad (40)$$

then the quantity

$$I = \frac{\partial L}{\partial \dot{x}^i} K^i + K \quad (41)$$

is a constant of motion. Since the family (39) of transformations is velocity-dependent, it is not generally considered a

family of diffeomorphisms. Nevertheless, it is a generalized symmetry of the action and Noether-related to an invariant of the form (41).

Conversely, if the quantity  $I$  is a constant of motion, then the  $\epsilon$ -family of transformations generated by  $K^i(x, \dot{x}, t)$ , obtained by solving the linear system (Ioannou & Apostolatos 2004)

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} K^i = \frac{\partial I}{\partial \dot{x}^j}, \quad (42)$$

is a generalized symmetry of the action.

The motion of a test particle in a Newtonian gravitational field can be obtained from the Lagrangian  $L(x, \dot{x}) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \Phi$ . If the motion admits a quadratic invariant  $I(x, \dot{x}) = K_{ij} \dot{x}^i \dot{x}^j + K$ , then the inverse Noether theorem (42) implies that the  $\epsilon$ -family of transformations (39) generated by  $K^i = K^{ij} \dot{x}^j$  is a generalized symmetry of the action. We infer, for the problem of the previous section with  $K^{ij}$  given by Eq. (27), that the action (1) is invariant under three families of infinitesimal transformations:

(i) The Killing-Stäckel tensor  $\varphi^i \varphi^j$  is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon \varphi^i \varphi^j \dot{x}^j = x^i + \epsilon L_z \varphi^i$$

The tensor  $\varphi^i \varphi^j$  is of course reducible to the axial Killing vector  $\varphi^i$ , related to the family of diffeomorphisms  $x_\epsilon^i = x^i + \epsilon \varphi^i$ . These represent azimuthal rotations and give rise to conservation of angular momentum (7).

(ii) The metric  $g^{ij}$  is a Killing-Stäckel tensor and is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon g^{ij} \dot{x}^j = x^i + \epsilon \dot{x}^i$$

or, equivalently,

$$x_\epsilon^i(t) = x^i(t) + \epsilon \dot{x}^i(t) = x^i(t + \epsilon)$$

The metric tensor  $g^{ij}$  is therefore Noether-related to invariance with respect to time translations  $t \rightarrow t + \epsilon$  and gives rise to conservation of energy (or the Hamiltonian) given by Eq. (2).

(iii) The irreducible Killing-Stäckel tensor  $A^{ij}$  given by eq. (28) is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon A^{ij} \dot{x}^j$$

or, equivalently,

$$\begin{aligned} \mathbf{x}_\epsilon &= \mathbf{x} + \epsilon [(\mathbf{x} \times \dot{\mathbf{x}}) \times \mathbf{x} - (\mathbf{a} \times \dot{\mathbf{x}}) \times \mathbf{a}] \\ &= \mathbf{x} + \epsilon [(\mathbf{x} - \mathbf{a}) \times \dot{\mathbf{x}}] \times (\mathbf{x} + \mathbf{a}) \end{aligned}$$

This a-posteriori knowledge of the symmetry transformation allows a fast “derivation” of the quadratic invariant via the Noether procedure: varying the action of the two-center problem with respect to the above family of transformations yields no change, while the Lagrangian changes by  $-\epsilon dA/dt$  with  $A$  given by eq. (38). Then, Eq. (41) leads immediately to the quadratic invariant (36).

As mentioned above, these transformations are a symmetry of the action, giving rise to the constant of motion (35), but are not diffeomorphisms since they depend on position and velocity. Nevertheless, writing  $\epsilon \dot{x}^j(t) = x^j(t + \epsilon) - x^j(t)$  they can be expressed as transformations in position and time:

$$x_\epsilon^i(t) = x^i(t) - A^{ij} x_j(t) + A^{ij} x_j(t + \epsilon)$$

where  $A^{ij}$  is a function of  $x^i(t)$ .

The Noether theorem and its inverse, expressed by Eqs. (39)–(42), also hold for a relativistic Lagrangian and can be used to show that the four constants of geodesic motion in a Kerr (or Kerr-de Sitter) spacetime are also Noether-related to symmetries of the action. Axisymmetry is related to conservation of angular momentum about the symmetry axis as in case (i) above. Case (ii) discussed above has two analogues in general relativity: Stationarity (invariance of the Lagrangian under time translations along the integral curves of a timelike Killing vector) is related to conservation of energy or Hamiltonian. Metric affinity (invariance of the action under proper time translations) is related to conservation of the magnitude of four-velocity (or the super-Hamiltonian) and is associated with the four-metric being a Killing tensor. Finally, a family of transformations analogous to those of case (iii) is related to the Carter constant of motion (c.f. Padmanabhan 2010, page 381).

### 3 INVARIANTS QUARTIC IN THE MOMENTA

#### 3.1 Killing equation and integrability conditions

The rank-two Killing-Stäckel tensor (28) of the previous section reduces to a symmetrized combination of Killing vectors in the spherically symmetric limit  $a \rightarrow 0$ . One might expect the next natural generalization to be a rank-four Killing-Stäckel tensor, which would reduce to a combination of lower order tensors in some appropriate limit. We thus consider invariants quartic in the momenta

$$I(x, p) = K^{ijkl}(x)p_i p_j p_k p_l + K^{ij}(x)p_i p_j + K(x) \quad (43)$$

associated with a rank-four Killing-Stäckel tensor  $K^{ijkl}$ . The above quantity is conserved iff it commutes with the Hamiltonian, in the sense of a vanishing Poisson bracket, Eq. (9). Evaluating the bracket with  $I$  given by Eq. (43) and  $H$  given by Eq. (2) yields

$$\begin{aligned} \{I, H\} = & \partial^m K^{ijkl} p_i p_j p_k p_l p_m \\ & + (\partial^k K^{ij} - 4K^{ijkl} \partial_l \Phi) p_i p_j p_k \\ & + (\partial^k K - 2K^{ik} \partial_i \Phi) p_k \end{aligned} \quad (44)$$

Demanding strong integrability, that is, requiring that this Poisson bracket vanish for all orbits, we have

$$\partial^m K^{ijkl} = 0 \quad (45)$$

$$\partial^k K^{ij} = 4K^{ijkl} \partial_l \Phi \quad (46)$$

$$\partial^k K = 2K^{ik} \partial_i \Phi \quad (47)$$

As in the previous section, a solution to Eq. (47) exists only if the following integrability condition is satisfied:

$$\partial^i (K^{jk} \partial_k \Phi) = 0 \quad (48)$$

which is identical to condition (13). The above set of equations is employed in Hietarinta (1987) to find quartic invariants for various two-dimensional systems.

Eq. (46) may be regarded as an inhomogeneous Killing equation and is also subject to an integrability condition. The general solution to this equation and its integrability

condition are obtained in Appendix A. We find that a solution to Eq. (46) in  $\mathbb{E}^2$  exists only if the following integrability condition is satisfied:

$$\begin{aligned} \partial_{zzz} (K^{yyyi} \partial_i \Phi) - 3\partial_{zz} (K^{yyzi} \partial_i \Phi) = \\ \partial_{yy} (K^{zzzi} \partial_i \Phi) - 3\partial_{yyz} (K^{yzzi} \partial_i \Phi) \end{aligned} \quad (49)$$

where  $i$  is summed over the  $y$  and  $z$  components and  $\partial_{ij\dots k}$  is an abbreviation for  $\partial_i \partial_j \dots \partial_k$ . A generalization of the above condition to  $\mathbb{E}^n$  is straightforward and, as shown in Appendix A, given by

$$\begin{aligned} \partial_{nml} f_{ijk} - \partial_{nmi} f_{ljk} - \partial_{njl} f_{imk} + \partial_{nji} f_{lmk} \\ - \partial_{kml} f_{ijn} + \partial_{kmi} f_{ljn} + \partial_{kjl} f_{imn} - \partial_{kji} f_{lmn} = 0 \end{aligned} \quad (50)$$

where  $f^{ijk} = 4K^{ijkl} \partial_l \Phi$ . Eq. (49) can be recovered from the above expression by setting  $n = m = i = z$  and  $j = k = y$ . Eq. (49) will suffice for our present purposes, as we shall restrict attention to two-dimensional motion in section 3.3 and beyond. If, in addition to  $H$  and  $L_z$ , a third constant of motion exists for all initial conditions in three dimensions, then this constant ought to also be conserved in the special case of orbits with  $L_z = 0$  that lie on a meridional plane. That is, three dimensional motion is integrable only if meridional motion is integrable. We may thus set  $x = 0$  and study motion restricted on the meridional ( $y$ - $z$ ) plane first. If such motion is integrable, generalization to three dimensions is straightforward by virtue of axisymmetry.

Eqs. (45)–(49) suggest a systematic method for obtaining potentials admitting quartic invariants:

- (i) The tensor  $K^{ijkl}$  may be computed by solving Eq. (45) subject to the symmetries of the problem.
- (ii) With this tensor known, the integrability condition (49) provides a restriction on the Newtonian potential  $\Phi$ . Solving this condition yields a family of possibly integrable potentials.
- (iii) Given a family of potentials  $\Phi$  which satisfy the condition (49), one may obtain the tensor  $K^{ij}$  by solving the inhomogeneous Killing equation (46).
- (iv) With  $K^{ijkl}$  and  $K^{ij}$  known, Eq. (49) provides a further restriction on the potential  $\Phi$ .
- (v) If a family of potentials  $\Phi$  are found that satisfy the above integrability conditions, then one may obtain the scalar function  $K$  by integrating the components of Eq. (47).

We now proceed to carry out the above steps in more detail.

#### 3.2 Rank-four Killing-Stäckel tensors

In  $\mathbb{E}^3$ , Eq. (45) constitutes an overdetermined system of 21 equations for the 15 unknown independent components of the symmetric tensor  $K^{ijkl}$ . Similarly, in  $\mathbb{E}^2$ , Eq. (45) is a system of 6 equations for 5 unknowns. The system may be solved by noticing that each component of  $K^{ijkl}$  must be a fourth order polynomial in the Cartesian coordinates  $x^i$ . Horwood (2008) considered the equation  $\partial^{(j} K^{i_1 \dots i_N)} = 0$  for a tensor  $K^{i_1 \dots i_N}$  of arbitrary valence  $N$  and showed that the general solution is given by

$$\begin{aligned} K = \sum_{L=0}^N \binom{N}{L} c_L^{i_1 \dots i_L i_{L+1} \dots i_N} \mathbf{X}_{i_1} \otimes \dots \\ \dots \otimes \mathbf{X}_{i_L} \otimes \mathbf{R}_{i_{L+1}} \otimes \dots \otimes \mathbf{R}_{i_N} \end{aligned} \quad (51)$$

where the objects  $\mathcal{C}_L^{i_1 \dots i_N}$ , labelled by  $L = 0, \dots, N$ , are constant and subject to the symmetries  $\mathcal{C}_L^{i_1 \dots i_L j_{L+1} \dots j_N} = \mathcal{C}_L^{(i_1 \dots i_L)(j_{L+1} \dots j_N)}$ . The above expression, for  $N = 4$  and with  $\mathbf{X}_i, \mathbf{R}_j$  given by Eq. (3), provides the general solution to Eq. (45). Imposing known isometries on the quartic invariant (43) constrains the coefficients  $\mathcal{C}_L^{i_1 \dots i_N}$ .

Stationarity has already been imposed (as the system is autonomous). For three dimensional motion, axisymmetry may be imposed via the requirement that the above Killing-Stäckel tensor is Lie-derived by the axial vector  $\varphi = \mathbf{R}_z$ , that is

$$\begin{aligned} \mathcal{L}_\varphi K^{ijkl} &= \varphi^m \partial_m K^{ijkl} - K^{mjkl} \partial_m \varphi^i - K^{mikl} \partial_m \varphi^j \\ &\quad - K^{mijl} \partial_m \varphi^k - K^{mijk} \partial_m \varphi^l = 0 \end{aligned} \quad (52)$$

(For purely meridional motion, enforcing the above condition does not change the relevant coefficients of  $K^{ijkl}$  projected to the meridional plane.) In addition, we also require  $\mathbb{Z}_2$  reflection symmetry about the equatorial plane, that is, the replacement  $\{z, p_z\} \rightarrow \{-z, -p_z\}$  leaves the quantity  $K^{ijkl} p_i p_j p_k p_l$  unchanged. Since the latter quantity is a fourth order polynomial in the (Cartesian) position and momentum variables, imposing the above isometries is straightforward via algebraic manipulation software such as Mathematica. Imposing the above symmetries significantly reduces the number of independent polynomial coefficients. Then, after dropping trivial terms such as  $g^{(ij} g^{kl)}$ ,  $g^{(ij} \varphi^k \varphi^l)$ ,  $\varphi^i \varphi^j \varphi^k \varphi^l$  which correspond to reducible Killing-Stäckel tensors, we find that the most general nontrivial solution to (45) subject to the symmetry  $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$  can be written in the remarkably simple form

$$K^{ijkl} = A^{(ij} B^{kl)} \quad (53)$$

where  $A^{ij}$  is given by Eq. (28) and  $B^{ij}$  is given by the same equation with the replacement  $\mathbf{a} \rightarrow \mathbf{b}$ :

$$A^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + a^k)(x^l - a^l) \quad (54)$$

$$B^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + b^k)(x^l - b^l) \quad (55)$$

where  $\mathbf{a} = a \hat{\mathbf{z}}, \mathbf{b} = b \hat{\mathbf{z}}$ . Note that the above expressions are valid in two and three dimensions. This completes step (i) of our prescribed procedure.

### 3.3 First integrability condition: no-hair relation

With the tensor  $K^{ijkl}$  known, the next step consists of finding the class of gravitational potentials  $\Phi$  that satisfy the integrability condition (49). A stationary axisymmetric vacuum potential that vanishes at infinity is characterized by the multipole expansion (29). Assuming equatorial plane reflection symmetry, the odd moments  $M_L$  ( $L = 1, 3, \dots$ ) vanish. Restricting attention to purely meridional motion, with  $K^{ijkl}$  and  $\Phi$  given by eqs. (53) and (29), we find with straightforward algebra that the integrability condition (49) is satisfied if and only if the even multipole moments satisfy the two-step recursion relation

$$M_{L+4} = (a^2 + b^2)M_{L+2} - a^2 b^2 M_L \quad (L = 0, 2, \dots) \quad (56)$$

This equation generalizes the “no-hair” relation (30). One may obtain all higher moments recursively from the first two nonvanishing moments. This yields

$$M_L = \frac{(a^{2L} - a^L b^2)M_0 + (a^L - b^L)M_2}{a^2 - b^2} \quad (57)$$

for  $L$  even (and  $M_L = 0$  for  $L$  odd). In the limit  $b \rightarrow 0$ , one recovers the no-hair relations (30) and (31) (except for  $L = 0$ ). The limit  $b \rightarrow a$  is slightly more subtle. Although the  $b \rightarrow a$  limit of eq. (56) is satisfied by (30) the converse is not necessarily true, because Eq. (56) does not fix the relation between the first two moments. Applying Eq. (56) recursively after taking the limit  $b \rightarrow a$  (or taking the same limit of Eq. (57) and applying the l’Hôpital rule) yields

$$\lim_{b \rightarrow a} M_L = a^L \left[ M_0 + \frac{L}{2}(a^{-2}M_2 - M_0) \right] \quad (L = 0, 2, \dots) \quad (58)$$

If the mass and quadrupole moment are related by  $M_2 = a^2 M_0$  then one recovers Eq. (31), but this need not be the case and  $M_2$  is generally considered independent of  $M_0$ . If  $a \neq b$ , then the gravitational field depends on four independent parameters ( $M_0, M_2, a, b$ ) and the reparametrization  $m_a = (M_2 - b^2 M_0)/(a^2 - b^2)$ ,  $m_b = (M_2 - a^2 M_0)/(b^2 - a^2)$  allows us to write Eq. (57) in the suggestive form

$$M_L = a^L m_a + b^L m_b \quad (L = 0, 2, \dots) \quad (59)$$

Then, summing the Legendre series (29) yields the potential

$$\begin{aligned} \Phi = & -\frac{m_a/2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{m_a/2}{\sqrt{x^2 + y^2 + (z+a)^2}} \\ & -\frac{m_b/2}{\sqrt{x^2 + y^2 + (z-b)^2}} - \frac{m_b/2}{\sqrt{x^2 + y^2 + (z+b)^2}} \end{aligned} \quad (60)$$

created by four fixed point sources with mass  $m_a/2$  at positions  $\pm a \hat{\mathbf{z}}$  and mass  $m_b/2$  at positions  $\pm b \hat{\mathbf{z}}$ . Our solution procedure guarantees that the above expression gives the unique vacuum potential with the symmetry  $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$  compatible with the integrability condition (49). This completes step (ii) of the prescribed procedure.

### 3.4 Second integrability condition: nonexistence of quartic invariants

With  $\Phi$  given by Eq. (60), one may use eqs. (A6)-(A9) to solve the inhomogeneous Killing equation (46). We find

$$\begin{aligned} K^{ij} = & 2BA^{ij} + 2AB^{ij} \\ & + (b^2 - a^2)(\tilde{A} - \tilde{B})(z\delta_y^i \delta_y^j - y\delta_y^i \delta_z^j) + \nu C^{ij} \end{aligned} \quad (61)$$

where  $\nu$  is a constant,  $A$  is given by Eq. (35) with the replacement  $M_0 \rightarrow 2m_a$ ,  $\tilde{A}$  is given by

$$\tilde{A} = \frac{m_a(z+a)}{\sqrt{y^2 + (z+a)^2}} + \frac{m_a(z-a)}{\sqrt{y^2 + (z-a)^2}} \quad (62)$$

and  $B, \tilde{B}$  are given by Eqs. (35), (62) with the replacements  $a \rightarrow b$  and  $m_a \rightarrow m_b$ . The additive contribution  $C^{ij}$  to Eq. (61) is a solution to the homogenous Killing equation (11), which is polynomial of second order in the cartesian coordinates. Since the homogeneous solution must obey the symmetries of the problem, it must have the form of Eq. (28), with  $a$  replaced by some other parameter  $c$ , that is

$$C^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + c^k)(x^l - c^l) \quad (63)$$

where  $\mathbf{c} = c \hat{\mathbf{z}}$ . This completes step (iii) of our prescribed procedure.

The next step is to solve the second integrability condition (48). However, substituting Eqs. (60) and (61) into the condition (48), we find that the latter *cannot be satisfied*



except in the limit  $b \rightarrow a$ , whence the quartic invariant is reducible to the quadratic invariant of the previous section. We infer that *there exists no stationary, axisymmetric, equatorially symmetric vacuum Newtonian potential that admits an independent nontrivial invariant quartic in the momenta*. That is, the only quartic invariants are trivial products of lower-order invariants.

The proof of this nonexistence result uses the fact that strong integrability requires existence of a constant of motion for all values of energy  $E$  and angular momentum  $L_z$ , including the case of purely meridional orbits with  $L_z = 0$ . Failure to satisfy Eq. (49) means that there exists no independent constant for purely meridional motion; therefore there exists no strong integral for three-dimensional motion. One could alternatively use the integrability condition (50) to derive the same nonexistence result directly in  $\mathbb{E}^3$ .

The nonexistence of a quartic invariant does not preclude the possibility that the potential (60) admits some other constant of motion with different dependence in the momenta. A superficial study of Poincaré maps of three dimensional orbits in the four-center potential may show that most orbits appear to be regular. This could lead to the impression that the system is integrable. However, a thorough scan of initial conditions reveals the existence of Birkhoff chains and in some cases ergodic motion surrounds the main island of stability on the Poincaré maps, confirming the non-integrability of the system described by the potential (60). In a relativistic context, a similar behaviour has been observed for orbits in certain stationary axisymmetric spacetimes (Apostolatos et al. 2009; Lukes-Gerakopoulos et al. 2010; Contopoulos et al. 2011).

## 4 SUMMARY AND CONCLUSIONS

The aim of this work was to study polynomial constants of motion in stationary axisymmetric gravitational fields in a Newtonian context and to explore the analogies with similar problems in relativistic gravity. The results in this study that are (to the author's knowledge) new include:

- (i) the uniqueness of the constant (35)-(36) of the Euler problem: Eq. (32) gives the only stationary, axisymmetric, equatorially symmetric Newtonian vacuum potential admitting a Killing-Stäckel tensor of rank two;
- (ii) the uniqueness of the constant (36)-(38) of the Lagrange problem: Eq. (37) gives the only stationary, axisymmetric, equatorially symmetric Newtonian potential admitting a Killing-Stäckel tensor of rank two;
- (iii) the relation of the quadratic constant in the Lagrange problem to that of Lynden-Bell (2003), and the analogy with the Carter constant in a Kerr-de Sitter spacetime;
- (iv) the integrability conditions (49), (50) which do not appear to have been implemented previously in direct searches for quartic invariants (c.f. Hietarinta 1987).
- (v) the non-existence of stationary, axisymmetric, equatorially symmetric, vacuum Newtonian potentials admitting a Killing-Stäckel tensor of rank four.

Note that the assumptions of vacuum and equatorial symmetry may be relaxed and the two-center problem remains integrable (Lynden-Bell 2003). Note also that the integrability condition (49) and its generalization to arbitrary dimen-

sion, Eq. (50), can be quite useful for other direct searches of quartic invariants.

The methods employed here may provide intuition for treating similar problems in relativistic gravity or post-Newtonian approximations to it. This could lead to further insights on how to parametrize the departure of a spacetime geometry from that of a Kerr spacetime and the relation of this departure to integrability or ergodicity. Extrapolating the present results to the relativistic regime would lead to the conjecture that rank-4 Killing tensors do not exist in stationary axisymmetric vacuum spacetimes. This conjecture would contradict claims made by Brink (2011) on existence of rank-4 Killing tensors in certain stationary axisymmetric spacetimes, but would be in agreement with the nonexistence results of Kruglikov & Matveev (2011) and Lukes-Gerakopoulos (2012) in the case of the Zipoy-Voorhees spacetime. Nevertheless, extending the Newtonian analysis to generic stationary axisymmetric vacuum spacetimes is a nontrivial step.

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## APPENDIX A: INTEGRABILITY CONDITIONS AND INHOMOGENEOUS KILLING EQUATIONS

We consider the inhomogeneous Killing equation

$$\partial_{(k} K_{ij)} = f_{ijk} \quad (\text{A1})$$

where  $f^{ijk} = 4K^{ijkl}\partial_l\Phi$ . In  $\mathbb{E}^2$ , the cartesian components of the above equation read

$$\partial_y K_{yy} = f_{yyy} \quad (\text{A2})$$

$$\partial_z K_{yy} + 2\partial_y K_{yz} = 3f_{yyz} \quad (\text{A3})$$

$$\partial_y K_{zz} + 2\partial_z K_{zy} = 3f_{yzz} \quad (\text{A4})$$

$$\partial_z K_{zz} = f_{zzz} \quad (\text{A5})$$

The above system is overdetermined and may be solved as follows: Integrating Eq. (A2) with respect to  $y$  and Eq. (A5) with respect to  $z$  yields

$$K_{yy} = Z(z) + \int dy f_{yyy} \quad (\text{A6})$$

$$K_{zz} = Y(y) + \int dz f_{zzz} \quad (\text{A7})$$

where  $Z(z)$  and  $Y(y)$  are scalar functions of their arguments. Then, integrating Eq. (A3) with respect to  $y$  and Eq. (A4) with respect to  $z$  yields

$$K_{yz} = \zeta(z) + \frac{1}{2} \int dy (3f_{yyz} - \partial_z K_{yy}) \quad (\text{A8})$$

$$K_{zy} = \psi(y) + \frac{1}{2} \int dz (3f_{yzz} - \partial_y K_{zz}) \quad (\text{A9})$$

where  $\zeta(z)$  and  $\psi(y)$  are scalar functions of their arguments. Eqs. (A6)-(A9) provide the solution to the inhomogeneous Killing equation (A1).

Because  $K_{ij}$  is a symmetric tensor, the above two expressions must be equal. Acting with  $\partial_{yyzz} = \frac{\partial^4}{\partial y^2 \partial z^2}$  on Eqs. (A8), (A9), demanding that the two expressions be equal and using Eqs. (A6), (A7), yields the integrability condition

$$\partial_{zzz} f_{yyy} - 3\partial_{zz} f_{yyz} = \partial_{yy} f_{zzz} - 3\partial_{yy} f_{yzz} \quad (\text{A10})$$

which must necessarily be satisfied by  $f_{ijk}$  in order for Eq. (A1) to have a solution. Note that the unknown functions  $Z, Y, \zeta, \psi$  do not appear in the above condition. In the homogeneous case, these functions can be easily shown to be quadratic polynomials in their arguments by setting  $f_{ijk} = 0$ , demanding that expressions (A8) and (A9) be equal and separating variables.

The integrability condition (A10) may be generalized to  $\mathbb{E}^n$  as follows. Eq. (A1) can be expanded out as

$$\partial_i K_{jk} + \partial_j K_{ik} + \partial_k K_{ij} = 3f_{ijk} \quad (\text{A11})$$

The first term vanishes if we apply  $\partial_l$  and antisymmetrize over  $l$  and  $i$ . The second term vanishes if we subsequently apply  $\partial_m$  and antisymmetrize over  $m$  and  $j$ . Finally, the third term vanishes if we apply  $\partial_n$  and antisymmetrize over  $n$  and  $k$ . This yields the integrability condition

$$\begin{aligned} & \partial_{nml} f_{ijk} - \partial_{nmi} f_{ljk} - \partial_{njl} f_{imk} + \partial_{nji} f_{lmk} \\ & - \partial_{kml} f_{ijn} + \partial_{kmi} f_{ljn} + \partial_{kjl} f_{imn} - \partial_{kji} f_{lmn} = 0 \end{aligned} \quad (\text{A12})$$

which generalizes Eq. (A10) to arbitrary dimension.

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